Objectives:

- Apply knowledge of derivatives to rates of change in many fields.
- Practice challenging derivation techniques.

Note: These applications are only a few possible applications of the derivative. Be prepared to apply your general knowledge of derivatives to many different real-world applications.

Physics Application: Motion of an Object
Let $s(t)$ be a function that gives the position of a moving object at time t.

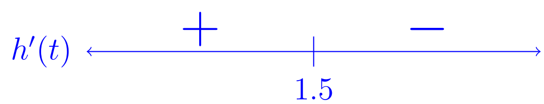
- The function that represents the velocity is $s'(t)$.
- The function that represents the acceleration is $s''(t)$.
- To find when an object is moving forward/upward, find intervals where $s'(t) > 0$.
- To find when an object is moving backward/downward, find intervals where $s'(t) < 0$.
- To find when an object is stationary, look for points where $s'(t) = 0$.
- Suppose $s(t)$ represents the height of a falling object.
  - To find the time when maximum height is achieved, we should
    (1) solve $s'(t) = 0$
    (2) plot the solutions for part (1) on a *labeled* number line
    (3) test points in each interval to determine the sign of $s'(t)$
    (4) identify points where the $s(t)$ is increasing on the left and decreasing on the right
  - To find the time of impact, we should solve $s(t) = 0$.
  - To find impact velocity, we should find time of impact and substitute that time into $s'(t)$.
- The object is speeding up when $s'(t)$ and $s''(t)$ have the same sign.
- The object is slowing down when $s'(t)$ and $s''(t)$ have opposite signs.

Physics Application:

A pumpkin spice muffin is launched upward from the roof of a building that is 64 feet high. The muffin is thrown with an initial velocity of 48 ft/sec. Its height in feet after t seconds is given by $h(t) = -16t^2 + 48t + 64$.

(a) On what interval(s) is the muffin moving upward?

The muffin is moving upward when the derivative is positive, so solve for $h'(t) > 0$.
Start by solving $h'(t) = 0$. $h'(t) = -32t + 48 = 0 \implies t = 1.5$
Draw a number line for $h'(t)$ and test points in each interval. Label the number line.



The derivative is positive to the left of $t = 1.5$, so the muffin is moving upward $\boxed{\text{on the interval } (0, 1.5)}$.

(b) When does the muffin reach maximum height? What is the maximum height?

The muffin's reaches maximum height at the top of the parabola, where $h'(t) = 0$. From our number line in part (a), we can see the $\boxed{\text{maximum height is achieved at } t = 1.5}$. To find the maximum height, plug $t = 1.5$ into the height function $h(t)$. So the max height is $h(1.5) = \boxed{100 \text{ ft}}$

(c) When does the muffin hit the ground? How fast is it going when it hits the ground?

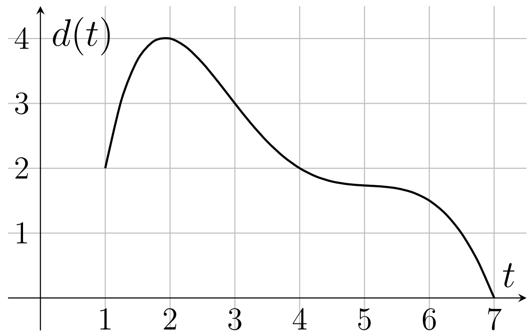
To find when the muffin hits the ground, we want to figure out when its height is zero. In other words, we solve for $h(t) = 0$.
$-16t^2 + 48x + 64 = 0 \implies -16(t^2 - 3t - 4) = 0 \implies -16(t-4)(t+1) = 0 \implies t = 4, -1$.
Since time must be a positive number, $\boxed{\text{the muffin must hit the ground at } t = 4}$. Its velocity when it hits the ground is $\boxed{h'(4) = -80 \text{ ft/sec}}$.

(d) On what interval(s) is the muffin speeding up?

We need $s'(t)$ and $s''(t)$ to solve this part. We found $s'(t)$ in part (a). Then $s''(t) = -32$, which is always negative. The muffin is speeding up when the signs of $s'(t)$ and $s''(t)$ are the same. Therefore the muffin in speeding up on $\boxed{(1.5, 4)}$. Note that the muffin stops speeding up at $t = 4$ because we determined in part (c) that this was when the muffin hit the ground.

Graphical problem:

Below is a graph of the position of an object in centimeters relative to some fixed point at t seconds.



Answer the following questions. Explain how you arrived at your answer.

(a) On what interval(s) is the object slowing down?

The object is slowing down when $d'(t)$ and $d''(t)$ have opposite signs. On the interval $(0, 2)$, the function is increasing but concave down, so $d'(t) > 0$ while $d''(t) < 0$. So on this interval, the function is slowing down. Below is an analysis of the whole graph.

interval	(0,2)	(2,3)	(3,5)	(5,7)
$d'(t)$	+	−	−	−
$d''(t)$	−	−	+	−
$d(t)$ is	slowing down	speeding up	slowing down	speeding up

The object is slowing down on $\boxed{(0, 2) \text{ and } (3, 5)}$.

(b) At what time(s) is the object stationary?

The object is stationary when its velocity is zero. In other words, the derivative is zero. The object is stationary at $\boxed{t = 2 \text{ and } t = 5}$.

(c) On what interval(s) is the object moving away from the fixed point?

The object is moving away from the fixed point when its distance from the fixed point is increasing, which means $d(t)$ is increasing. The object is moving away from the fixed point on the interval $\boxed{(1, 2)}$.

(d) At what time does the object reach its maximum speed?

The object reaches maximum speed when its derivative is the steepest. By inspecting the graph, the derivative seems steepest at $\boxed{t = 1}$.

Modeling Animal Populations:

The population of elk in a national park is modeled by $P(t) = \dfrac{2500}{1 + 24e^{-5t}}$, where t is time in decades since 1940.

(a) At $t = 0$, what is the population of the elk?

$P(0) = \dfrac{2500}{1 + 24e^{-5\cdot 0}} = \dfrac{2500}{1 + 24e^{0}} = \dfrac{2500}{25} = 100.$ There are $\boxed{\text{100 elk in 1940}}$.

(b) Calculate $P'(0)$. Give an interpretation of this quantity in the context of this problem. Remember to use units in your answer.

$P'(t) = -2500(1 + 24e^{-5t})^{-2} \cdot 24e^{-5t} \cdot (-5) = \dfrac{2500 \cdot 120e^{-5t}}{(1 + 24e^{-5t})^2}$

$P'(0) = \dfrac{2500 \cdot 120e^{0}}{(1 + 24e^{0})^2} = \boxed{480}$

$\boxed{\text{In the year 1940, the rate of growth of elk is 480 more elk per year}}$.

(c) What happens to the elk population in the long run? (Hint: To consider what happens in the long run, we are taking the *limit* of $P(t)$ as t goes to what value?)

This question is asking for the limit of $P(t)$ as t goes to infinity.

$\lim\limits_{t\to\infty} \dfrac{2500}{1 + 24e^{-5t}} = \dfrac{2500}{1 + 24 \cdot 0} = 2500$

In the long run, the population will $\boxed{\text{stabilize at around 2500 elk}}$.

In general, derivatives allow us to find rates of change. Think about your major or another topic you're interested in. What is something that could be modeled as a function? What would the derivative represent?

Additional Applications

Production Costs and Profit:

In an economics and business context, the derivative of cost is called the *marginal cost*. Suppose the cost in dollars of producing x bottles of organic chai is $c(x) = 1200 + 12x - 0.1x^2 + .0005x^3$.

(a) Find the function that represents marginal cost.

$\boxed{c'(x) = 12 - 0.2x + .0015x^2}$.

(b) What is the meaning of $c(200) = 3600$? Use units in your answer.

$\boxed{\text{The cost of producing 200 bottles of chai is \$3600}}$.

(c) What is the meaning of $c'(200) = 32$? Use units in your answer.

$\boxed{\text{After producing 200 bottles of chai, producing 1 more bottle of chai will cost \$32}}$.

(d) If each bottle of chai is sold for \$r, write an equation for the company's profit.

$P(x) = rx - c(x) = r(x) - \left(1200 + 12x - 0.1x^2 + .0005x^3\right) = -1200 + (r-12)x - 0.1x^2 + .0005x^3$

(e) If the company plans to make 200 bottles of chai, how much will they need to charge per bottle to make a \$1000 profit? If they charge this amount, what is the the derivative of profit (called marginal profit) for 200 bottles?

$$P(200) = 200r - 3600$$

So, if $P(200) = 1000$, then $r = 23$. They need to charge \$23 per bottle.

$$P'(x) = (r - 12) - 0.2x + .0015x^3 = 11 - 0.2x + .0015x^2$$

$$P'(200) = 31$$

So, at 200 bottles, producing one more bottle leads to \$31 more profit.

Computer Science:

Suppose you are using a sorting algorithm to reorder a list of stored data items. The time in nanoseconds to sort n entries is given by $T(n) = 234n\log_2(n)$.

(a) Write an equation for the rate of change in sorting time with respect to number of list items.

$$T'(n) = 234n\left(\frac{1}{\ln(2)n}\right) + 234\log_2(n) = \frac{234}{\ln(2)} + 234\log_2(n)$$

(b) If you have a 1,000 data items, how much will the sorting time increase if you add an additional item?

$$T'(1,000) = \frac{234}{\ln(2)} + 234\log_2(1,000) \approx 2,669.584\text{nanoseconds}$$

(c) If another sorting algorithm takes $f(n)$ nanoseconds where $f(n) = 121n^2$, which algorithm has a greater sorting time for 1,000 data items? Which algorithm has a greater increase in sorting time if an additional item is added to those 1,000 items?

$T(1,000) = 234(1,000)\log_2(1,000) \approx 2,331,993$ and $f(n) = 121,000,000$ so the first algorithm takes less time.

$f'(1000) = 242(1,000) = 242,000$ so the first algorithm also has a much smaller increase in sorting time when moving from 1,000 to 1,001 list items.

Bonus: One nanosecond is equal to one billionth of a second. Convert your solutions to units of seconds instead of nanoseconds.

Bacterial Growth:

A bacterial culture starts with 1200 bacteria in a dish of nutrient gel. The number bacteria after t minutes is $B(t) = 1200e^{0.04t}$.

1. Give a formula for the rate of change of the bacteria population in terms of t.

   $\boxed{B'(t) = 1200e^{0.04t} \cdot 0.04 = 48e^{0.04t}}$

2. Is the bacterial population growing faster at $t = 1$ or $t = 2$? Explain your answer.

   The rate of change of the bacterial population is $B'(t)$, so this question is asking whether $B'(1)$ or $B'(2)$ is bigger. Since $B'(2) > B'(1)$, the bacterial population is growing faster $\boxed{\text{at } t = 2}$.

Motion of a Spring:

The function $y = A\sin\left(\left(\sqrt{\frac{k}{m}}\right)t\right)$ represents the oscillations of a mass m at the end of a spring. The constant k measures the stiffness of the spring.

(a) Write an equation for the velocity of the mass at time t.

$$v(t) = A\cos\left(\left(\sqrt{\frac{k}{m}}\right)t\right)\sqrt{\frac{k}{m}}$$

(b) Find a value of t where the mass has velocity 0.

$v(t) = 0$ when $\cos\left(\left(\sqrt{\frac{k}{m}}\right)t\right) = 0$, or when $\frac{\sqrt{k}}{\sqrt{m}}t = \pm\frac{\pi}{2} \pm c\pi$ for some integer c. So, $t = \frac{\sqrt{m}}{\sqrt{k}}\left(\frac{\pi}{2} \pm c\pi\right)$. One value would be $t = \frac{\sqrt{m}\pi}{2\sqrt{k}}$.

(c) What is the period, T of the oscillation?

From graph transformations, we know the period is $2\pi\left(\frac{1}{\sqrt{\frac{k}{m}}}\right) = \frac{\sqrt{m}}{\sqrt{k}}2\pi$

(d) Find $\frac{dT}{dm}$. What does the sign of $\frac{dT}{dm}$ tell you?

$$\frac{dT}{dm} = \frac{2\pi}{\sqrt{k}}\left(\frac{1}{2}m^{-1/2}\right) = \frac{\pi}{\sqrt{km}}$$

Since this derivative is positive, we know increasing the mass of the object makes the oscillation period longer.

Measuring pH:

To compare the acidity of different solutions, chemists use the pH (which is a single number, not the product of p and H). The pH is defined in terms of the concentration, x, of hydrogen ions in the solution as $\text{pH} = -\log_{10}(x)$.

(a) Find the rate of change of pH with respect to hydrogen ion concentration when the pH is 2.

$$(\text{pH})' = -\left(\frac{1}{\ln(10)x}\right)$$

When the pH is 2, what is x? Well, then $-2 = \log_{10}(x)$ so $x = 10^{-2} = 0.01$. So the rate of change when pH=2 is $\frac{-1}{\ln(10)(0.01)} \approx -43.429$

(b) Suppose the concentration of hydrogen in a solution is equal to $h(t) = \frac{1}{10}(\sin(t)) + .5$ where t is time in hours.

(a) Write an equation for the pH of the solution as a function of t.

$$P(t) = -\log_{10}(h(t)) = -\log_{10}\left(\frac{1}{10}(\sin(t)) + .5\right)$$

(b) Write an equation for the rate of change of the pH with respect to time. After one hour is the pH of the solution increasing or decreasing? What about after 3 hours?

$$P'(t) = -\left(\frac{1}{\ln(10)\left(\frac{1}{10}\sin(t) + .5\right)}\right)\left(\frac{1}{10}\cos(t)\right)$$

$$P'(1) \approx 0.0402 \text{ and } P'(3) \approx -0.0836$$

So the pH is increasing after 1 hour and decreasing after 3 hours.

SCUBA Diving:

When SCUBA diving, it's (very) important to have an accurate estimate of how much air you will consume. Air consumption depends on the volume of air in the tank - if the same amount of air takes up less volume, it will be consumed more quickly.

(a) Boyle's Law states that the volume of air in cubic feet, $V(P)$ is given by $V(P) = \frac{P_0 V_0}{P}$ where P is the pressure on the tank in *psi* and P_0, V_0 are initial measurements of pressure and volume. A SCUBA tank of 80 cubic feet is typically filled to a pressure of 3,000 *psi*. Use these intitial measurements to write an equation for volume, V in terms of pressure, P.

$$V(P) = \frac{(3,000)(80)}{P} = \frac{240,000}{P}$$

(b) If d is the depth a diver in meters, the pressure on the air tank, $P(d)$ is given by $P(d) \approx P_0 + 1.368d$. Use this to write an equation for volume of air in the tank as a function of depth, $V(d) = V(P(d))$.

$$V(d) = \frac{240,000}{P(d)} = \frac{240,000}{3,000 + 1.368d} = 240,000\,(3,000 + 1.36d)^{-1}$$

(c) Find an equation for $V'(d)$. Include units. What does this represent?

$$V'(d) = -240,000\,(3,000 + 1.36d)^{-2}\,(1.36) = \frac{-328320}{(3,000 + 1.36d)^2}\frac{\text{ft}^3}{\text{meters}}$$

This derivative represents the increase/decrease in pressure as the depth increases or decreases.

(d) When is $V'(d)$ positive or negative? What does this mean about the effect of depth on air volume?

$V'(d)$ is always negative. So, increasing depth always decreases air volume.

Health:

Suppose you conduct a study in which you measure the effect of a particular training method on the time it takes an individual to run a mile.

(a) The control group did not participate in the special training, but did exercise on their own. The study found that for a person in the control group, their mile time in minutes was approximated by $f(t)$ where t is the number of hours per week the individual exercised. $f(t) = \dfrac{40}{x+5} + 5$.

1. Find $f'(t)$. Use this to find $f'(6)$ and explain what it represents.

$$f'(t) = -40(x+5)^{-2}$$

$$f'(6) = \frac{-40}{(6+5)^2} = \frac{-40}{121} \approx -0.33057$$

So, if a control group member is exercising 6 hours a week, they could lower their time by 0.33 minutes by exercising for one additional hour per week.

2. Find $\lim_{t\to\infty} f(t)$ and explain what it represents.

$$\lim_{t\to\infty} f(t) = 0 + 5 = 5$$

This means that control group members will see their time stabilize around 5 minutes.

(b) The experimental group did participate in the special training. The study found that for a person in the experimental group, their mile time in minutes was approximated by $g(t)$ where t is the number of hours per week the individual trained. $g(t) = \dfrac{100}{\left(\sqrt{x+5}\right)^3} + 4$.

1. Find $g'(t)$. Use this to find $f'(4)$ and explain what it represents.

$$g'(t) = 100\left(\frac{-3}{2}\right)(x+5)^{-5/2}$$

$$g'(4) = \frac{-300}{2\left(\sqrt{4+5}\right)^5} = \frac{-300}{486} \approx -0.617$$

So, those who are getting special training who train 4 hours a week can reduce their time by 0.617 minutes by training for one additional hour.

2. Find $\lim_{t\to\infty} g(t)$ and explain what it represents.

$$\lim_{t\to\infty} g(t) = 4$$

This means that experimental group members will see their time stabilize around 4 minutes.

(c) Does the training method seem to make a difference? (If in doubt, graph!)

Yes. We could compare in lots of ways, but overall the limits show you can achieve a lower time with the special training.